Probabilistic Graphical Models

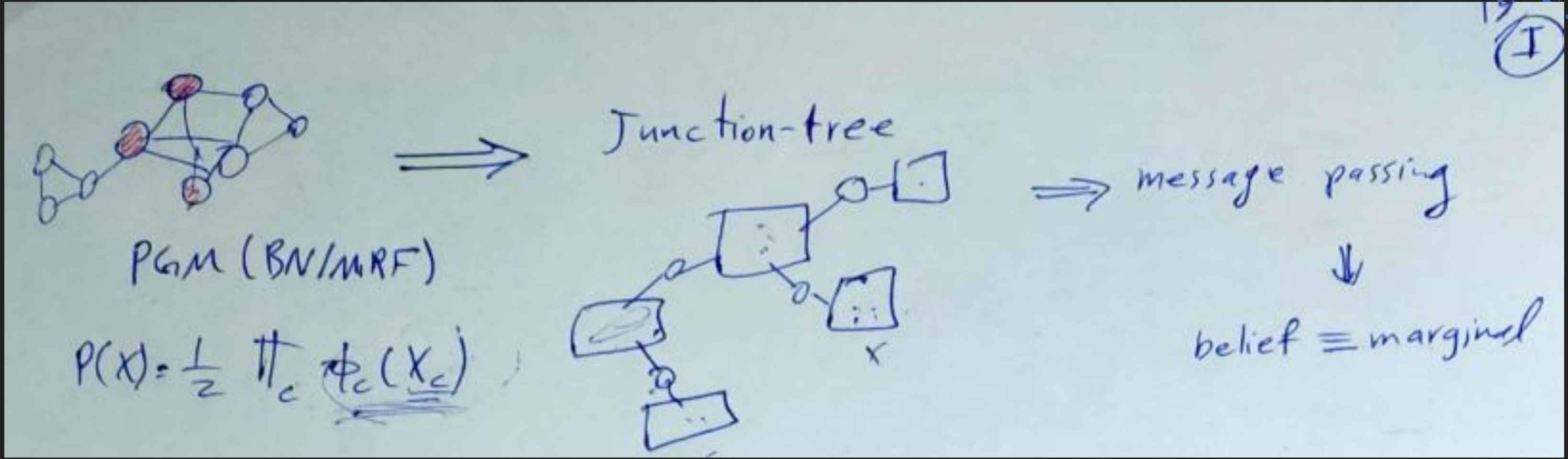
Lectures 14

Loopy Belief Propagation

Inference

- Exact
  - Variable Elimination
  - Message Passing - Junction Tree
  - Graph-cuts
- Approximate
  - Message Passing - Loopy Belief Propagation
  - Graph-cut based
  - Sampling Based
  - Variational Inference

How to build a cluster tree?



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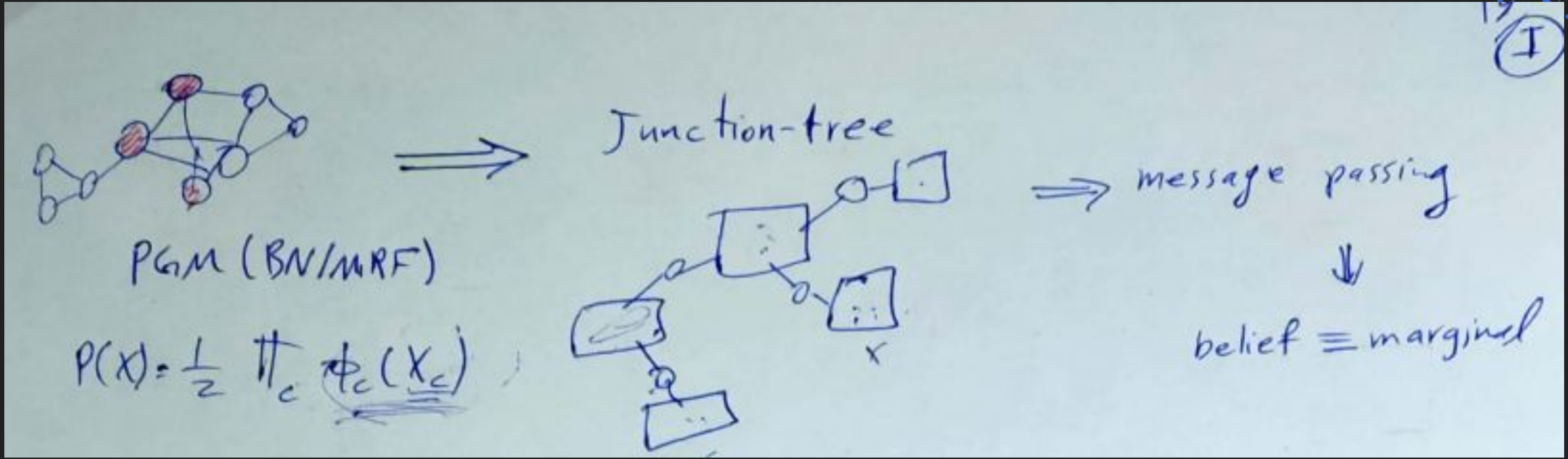

1. Using variable elimination

Build cluster tree using VE

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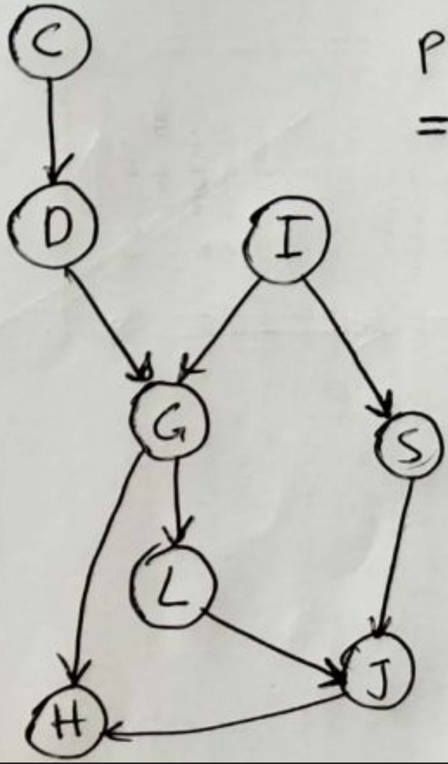
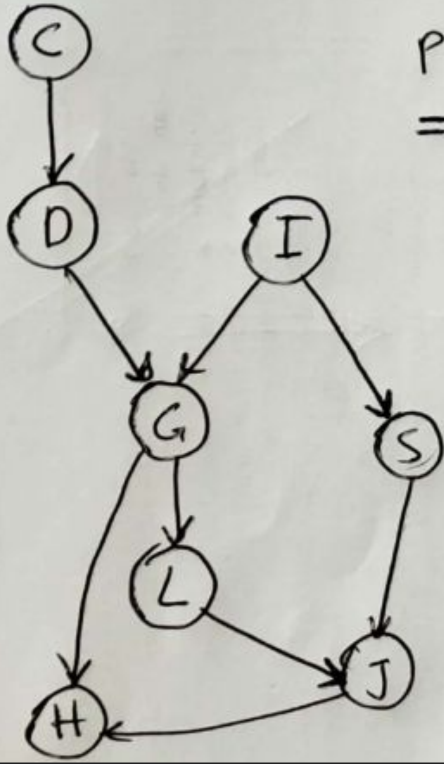


$$P(C,D,I,G,L,S,J,H)$$
$$= P(C)\, P(D|C)\, P(I)\, P(G|D,I)\, P(S|I)\, P(L|G)\, P(J|S,L)\, P(H|J,G)$$
$$= \boxed{\phi_1(C)\,\phi_2(C,D)}\ \boxed{\phi_3(I)}\ \boxed{\phi_4(G,D,I)}\ \boxed{\phi_5(S,I)}\ \boxed{\phi_6(L,G)}\ \boxed{\phi_7(J,S,L)}\ \boxed{\phi_8(H,J,G)}$$

Eliminate: $C, D, I, H, G, \underline{S}, \cancel{L}\, J$

Build cluster tree using VE



$$P(C,D,I,G,L,S,J,H)$$
$$= P(C)\,P(D|C)\,P(I)\,P(G|D,I)\,P(S|I)\,P(L|G)\,P(J|S,L)\,P(H|J,G)$$
$$= \boxed{\phi_1(C)\,\phi_2(C,D)}\,\boxed{\phi_3(I)}\,\boxed{\phi_4(G,D,I)}\,\boxed{\phi_5(S,I)}\,\boxed{\phi_6(L,G)}\,\boxed{\phi_7(J,S,L)}\,\boxed{\phi_8(H,J,G)}$$

Eliminate: C, D, I, H, G, S, ~~L~~ J

Build cluster tree using VE

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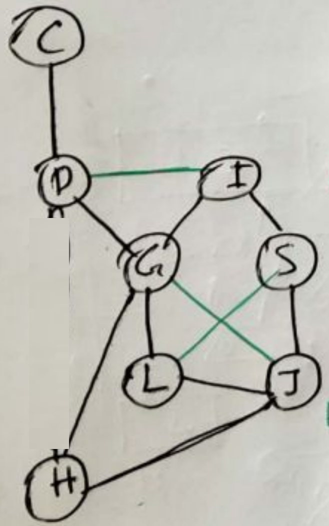

Eliminate C: $\tau_1(D) = \sum_{C} \phi_1(C)\,\phi_2(C,D)$

$\boxed{1: C, D} \xrightarrow[\tau_1(D)]{D}$

Eliminate D: $\tau_2(G,I) = \sum_{D} \phi_4(G,D,I)\,\tau_1(D)$

$\boxed{1: C, D} \xrightarrow[\tau_1(D)]{D} \boxed{D, G, I} \xrightarrow[\tau_2(G,I)]{G, I}$

Eliminate I:

$\cancel{\phi_4(G,D,I)}$

$\tau_3(S,G) = \sum_{I} \phi_3(I)\,\phi_5(S,I)\,\tau_2(G,I)$

Eliminate H:

Build cluster tree using VE

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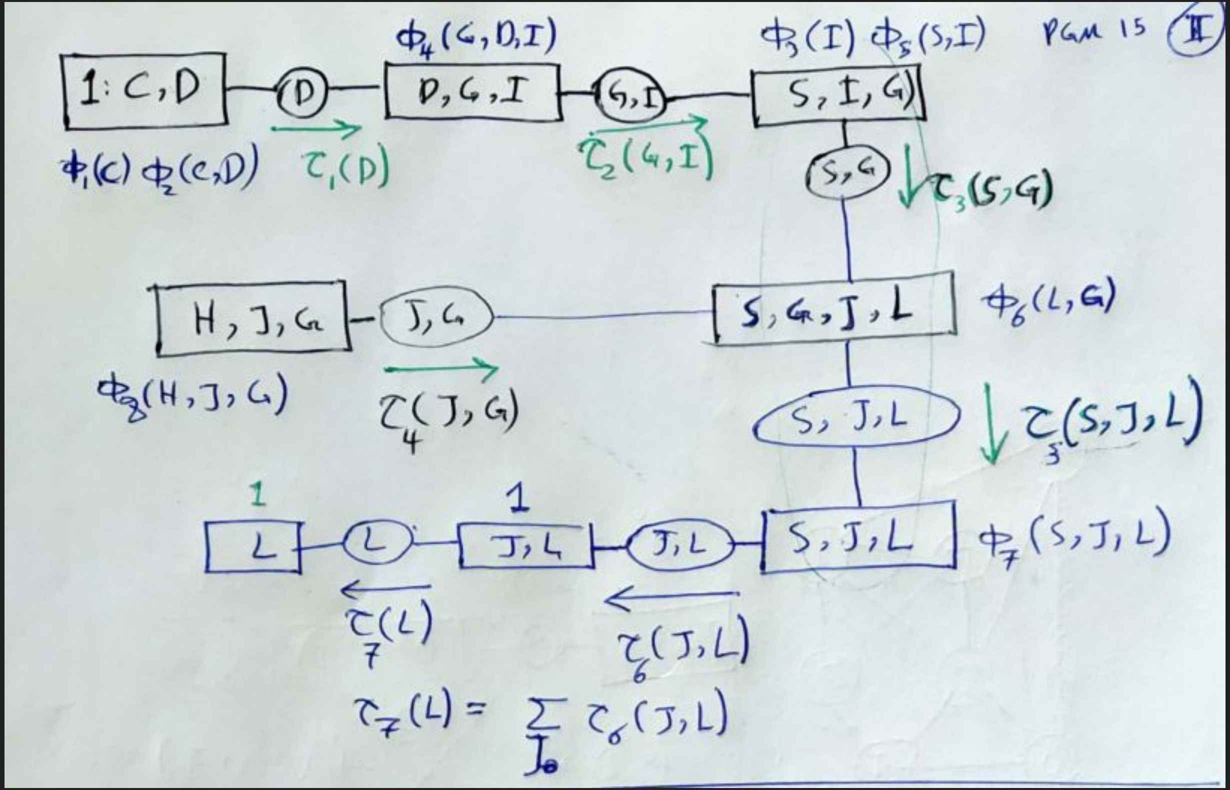


How to build a cluster tree?

1. Using variable elimination
2. Determine valid cluster trees

How to build a cluster tree?

Must possess tw properties

1- Family Preservation

Each Factor can be assigned to some cluster.

for each factor $\phi_c(X_c)$ there must be a cluster C_i with variables X_{C_i} such that $X_c \subseteq X_{C_i}$

2- Running Intersection Property

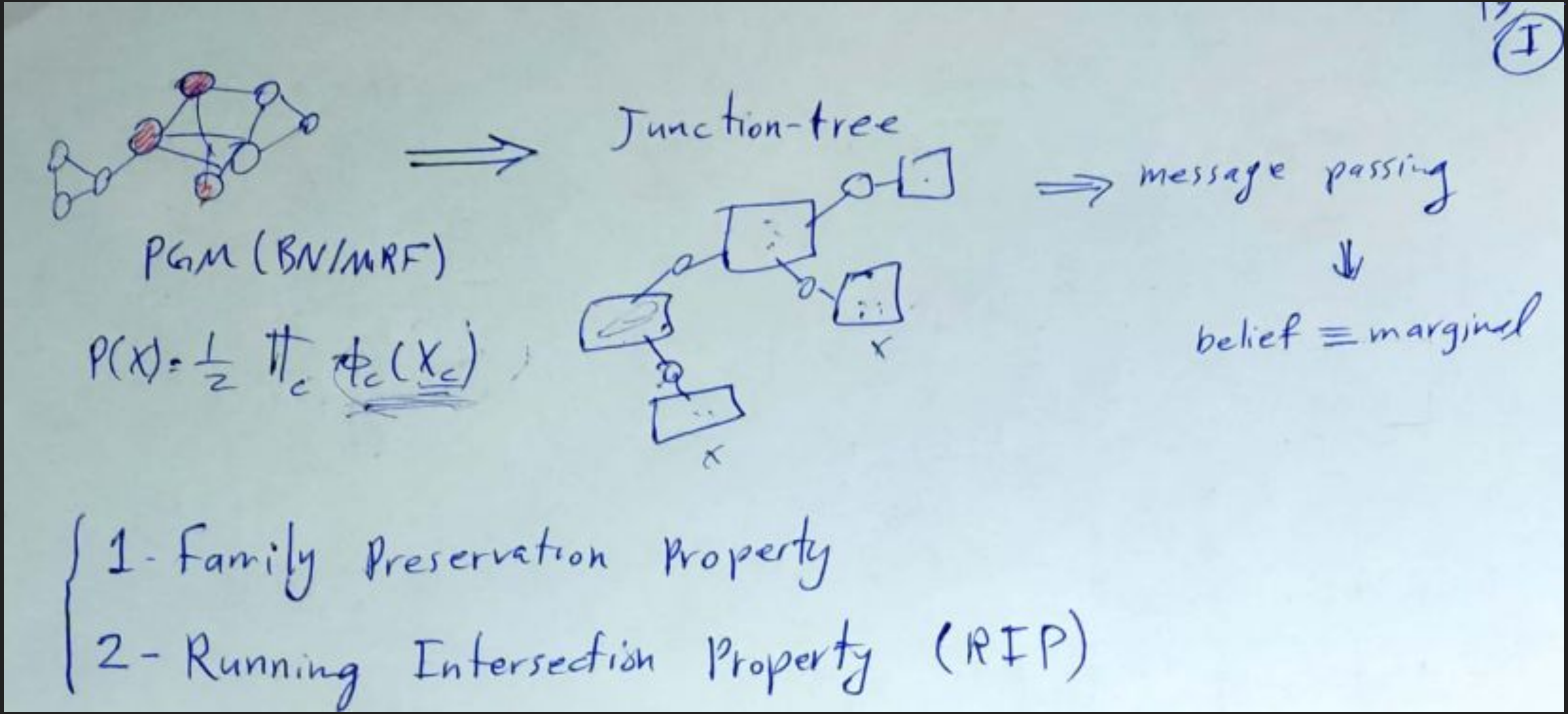
For each variable X the subgraph ~~contaiing~~ of the cluster-tree containing X ~~form~~ is connected (≡ also forms a tree).

How to build a cluster tree?

Must possess two properties

1. Family Preservation property: Each factor can be assigned to some cluster.
   a. For each factor $\varphi_I(X_I)$ there is a cluster C_i such that $X_I \subseteq X_{Ci}$.
2. Running Intersection Property: For each variable X the subgraph containing X is connected (and thus forms a tree).

How to build a cluster tree?



How to build a cluster tree?

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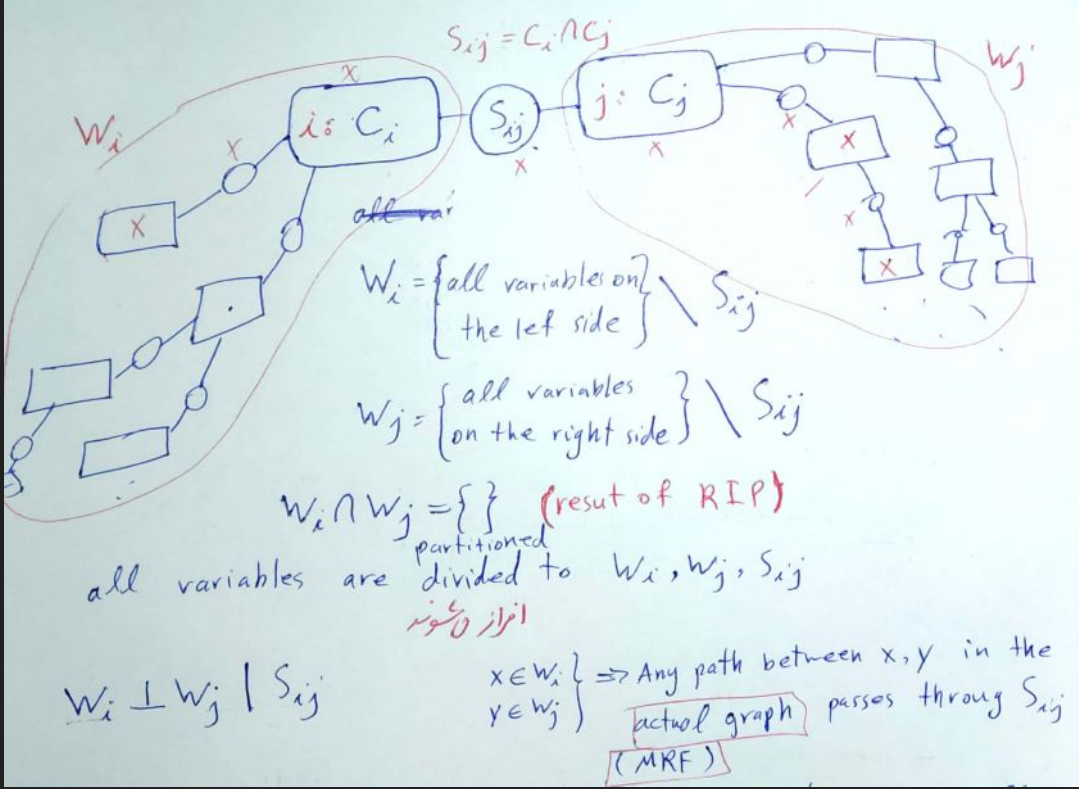
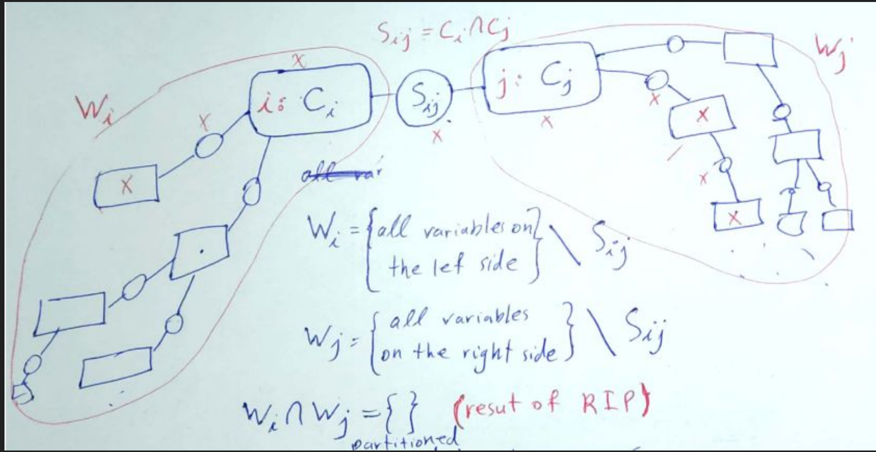
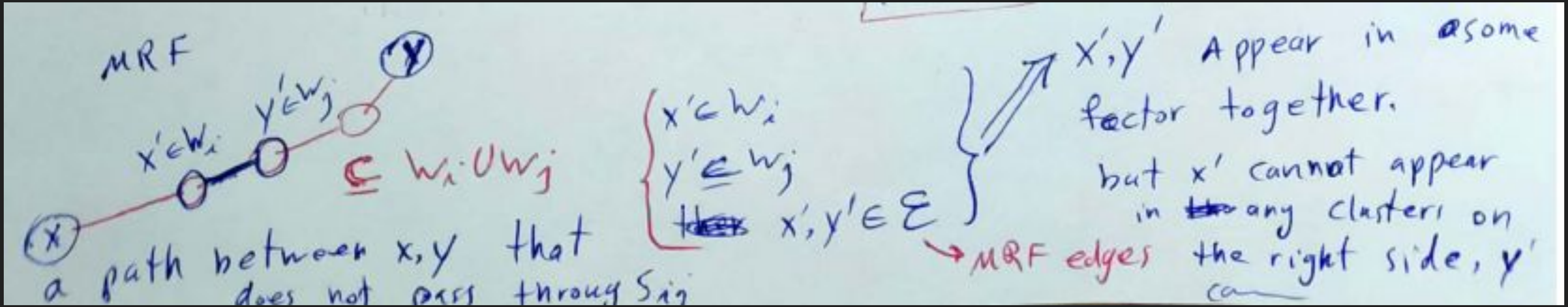
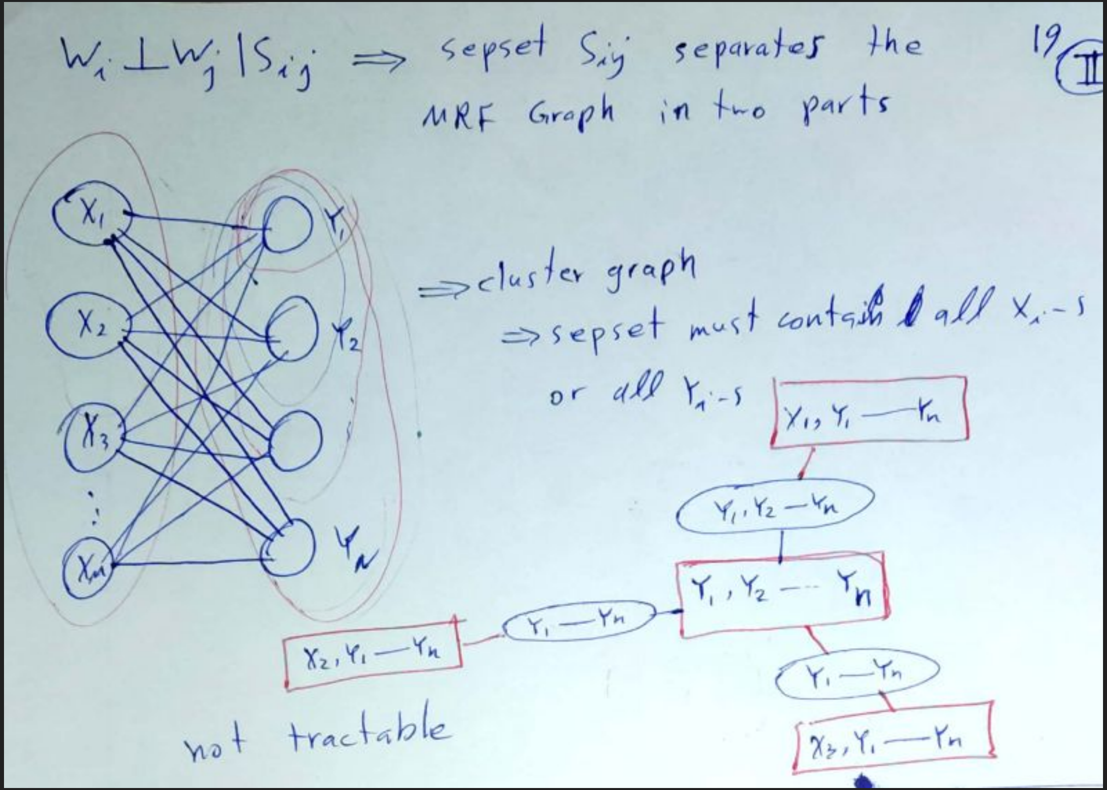


How to build a cluster tree?

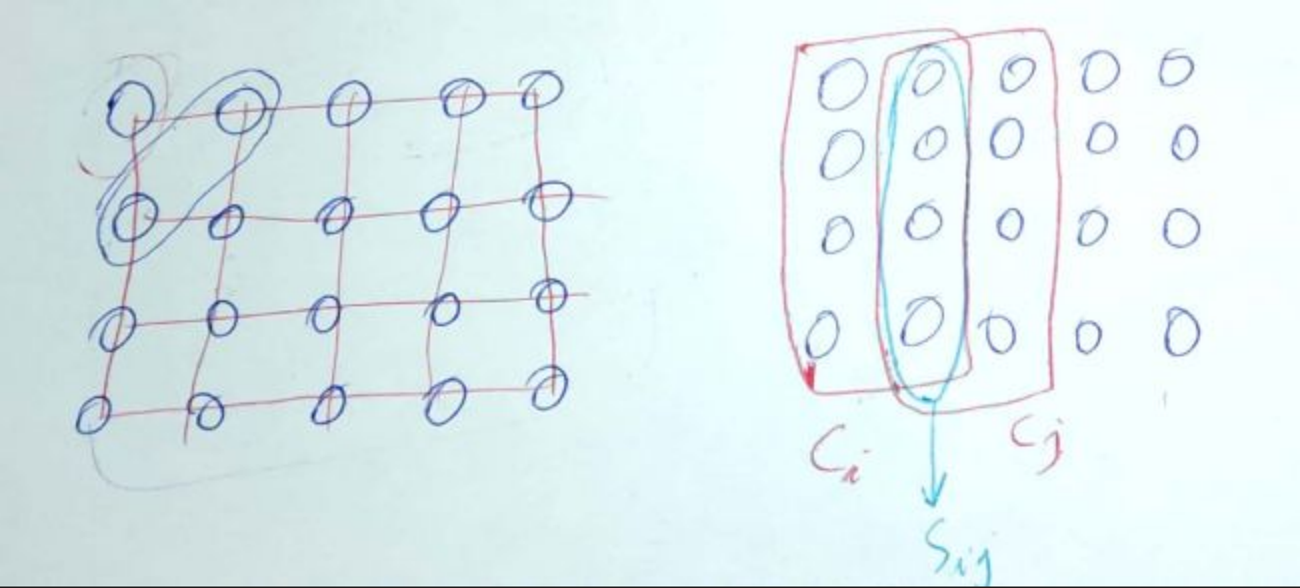




Junction tree - limitations



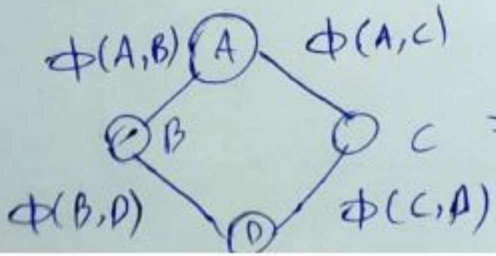
Junction tree - limitations



Loopy belief propagation

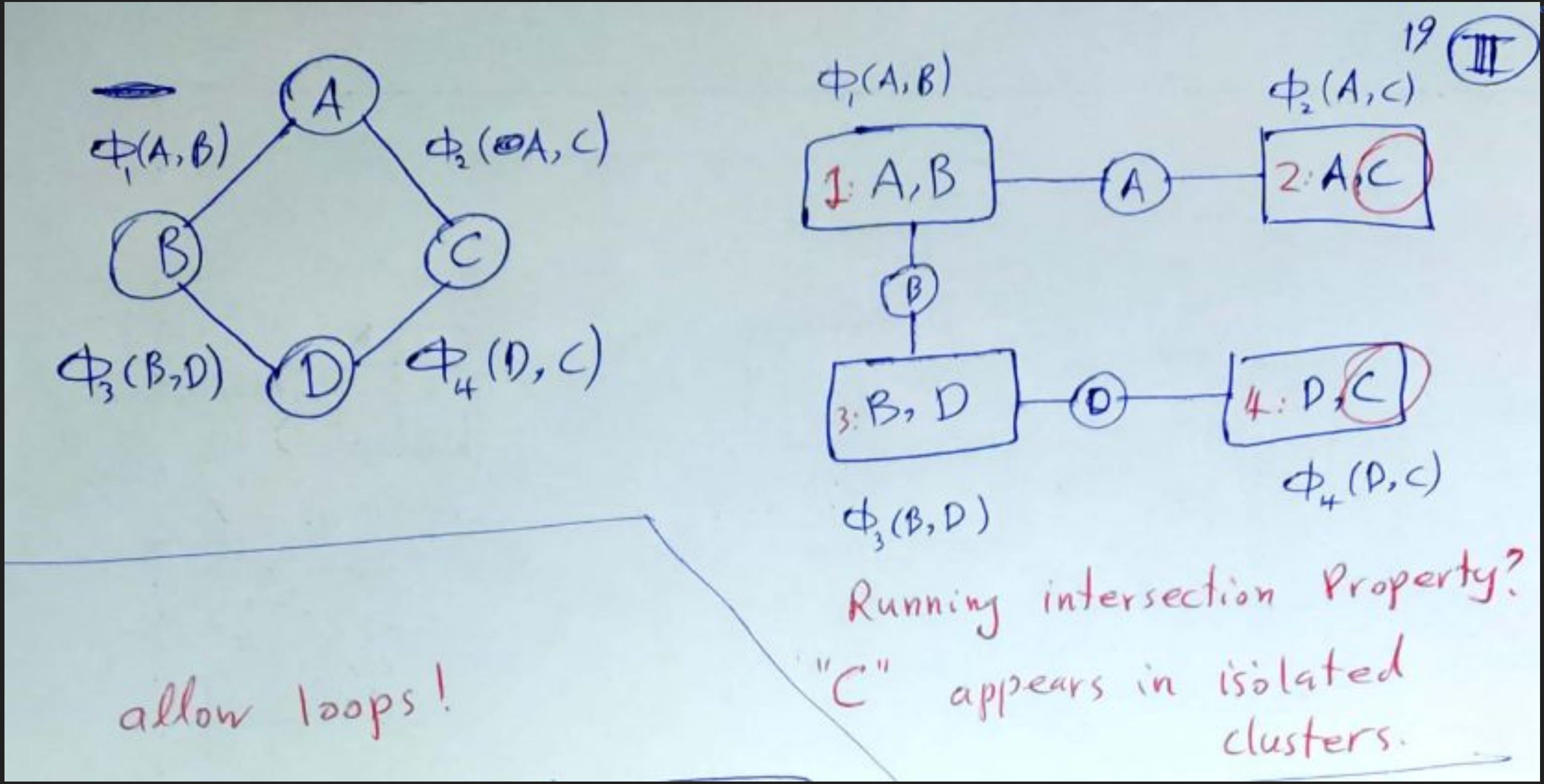
⇒ Approximate
- $\boxed{\text{Loopy Belief Propagation}}$
- Variational Inference
- Sample-Based

Loopy Belief Propagation: Allow the cluster graph to have loops

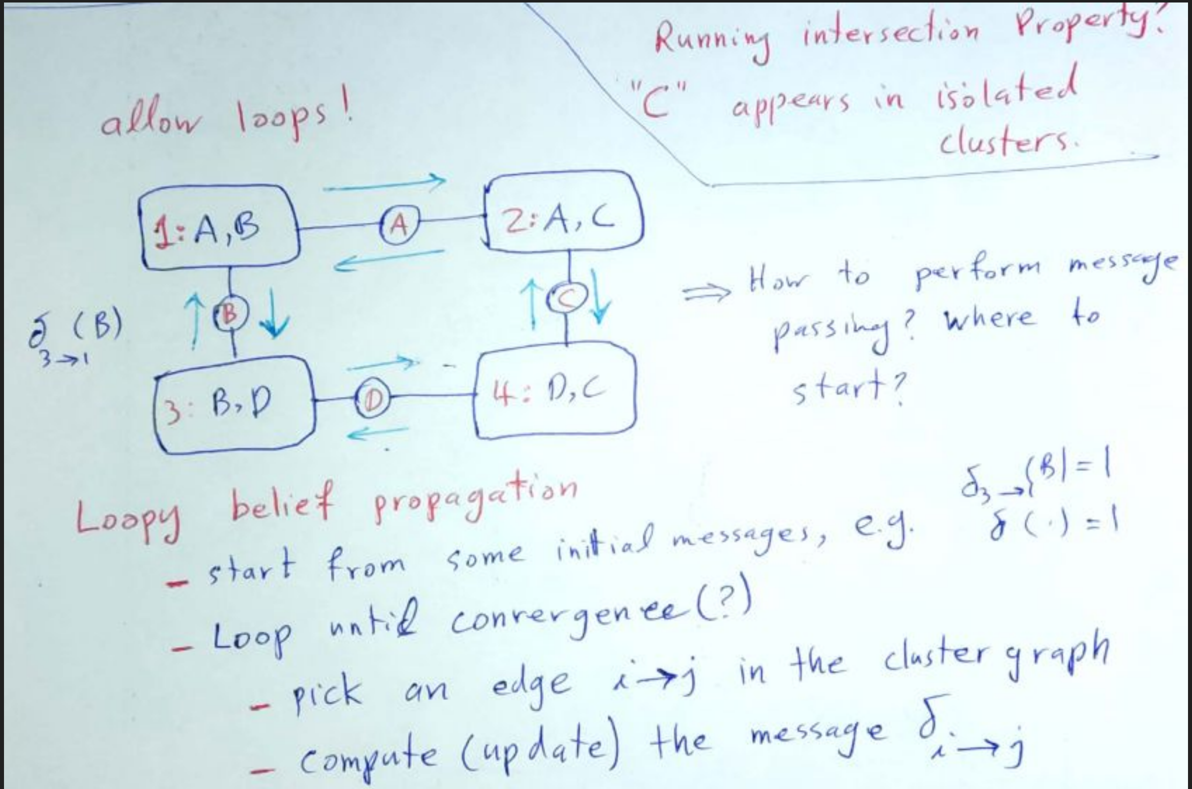


⇒ Can buidd a junction tree with each cluster having at most 2 ~~variable~~ variables?

Loopy belief propagation



Loopy belief propagation



Loopy belief propagation

Loopy belief propagation $\delta_{3\to 1}(B) = 1$

- start from some initial messages, e.g. $\delta(\cdot) = 1$
- Loop until convergence (?)
  - pick an edge $i \to j$ in the cluster graph
  - compute (update) the message $\delta_{i \to j}$

Messages are computed (updated) more than once

Does the algorithm converge? Not Always

Does it converge to the right solution? Usually Not

Does it give good approximate solutions? why?
Usually gives reasonable solutions

Loopy belief propagation

Cluster graph Properties ~~for~~

IV

1- Family Preservation (same as befor)

2- Running Intersection Property

if X appears in clusters C_i & C_j

$\Rightarrow$ there must be a path ~~where~~ between C_i and C_j such that X appears in all clusters and sepsets along the path

$\Rightarrow$ such a path must be unique.

Loopy belief propagation

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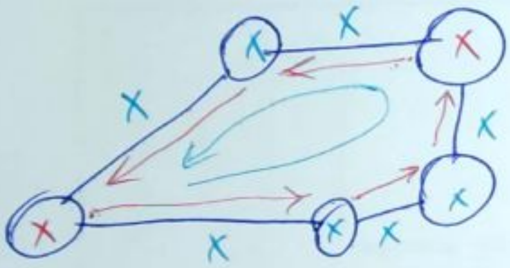
for each variable X the subgraph of the cluster graph containing X forms a tree.

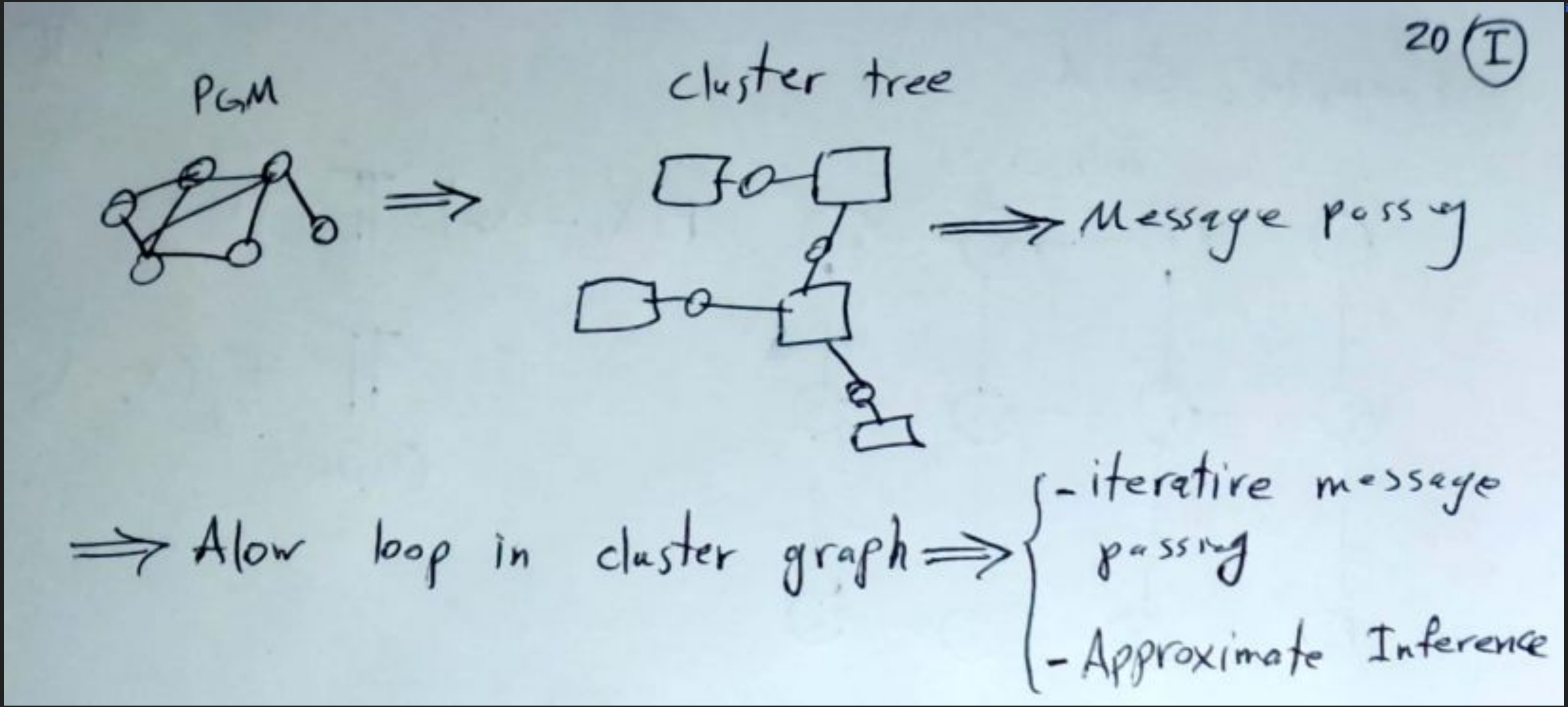


~~preve~~ trying to prevent falsely reinforcing a belief

In cluster graphs $S_{ij} \subseteq C_i \cap C_j$

In Cluster tree $S_{ij} = C_i \cap C_j$

Loopy belief propagation



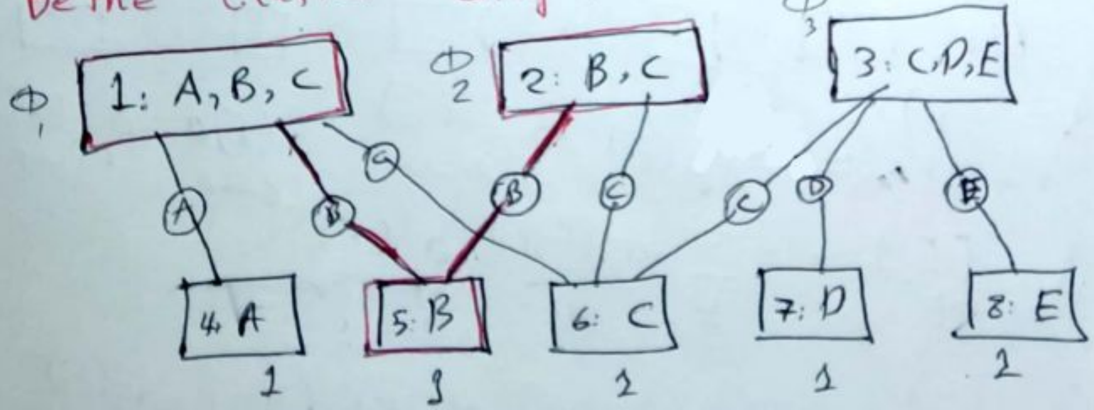
Is there at least one cluster graph?

Is there at least one cluster graph?

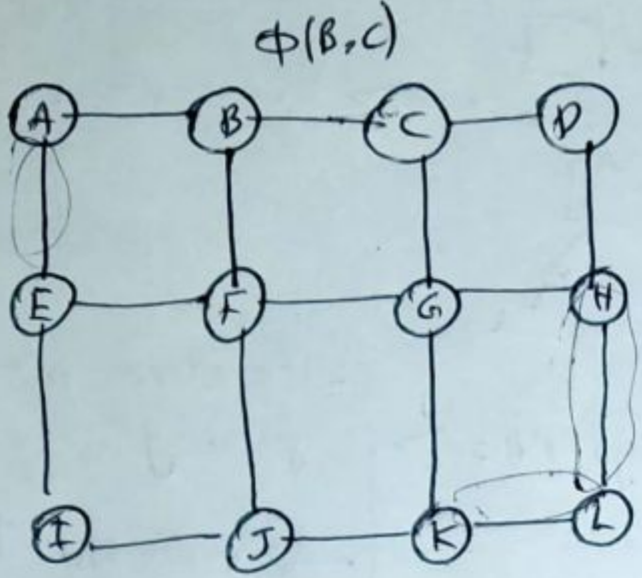
$P(A,B,C,D,E) = \Phi_1(A,B,C)\,\Phi_2(B,C)\,\Phi_3(C,D,E)$

is there a cluster graph with FPP & RIP such that the cluster sizes are at most the size of factors?

Bethe Cluster Graph

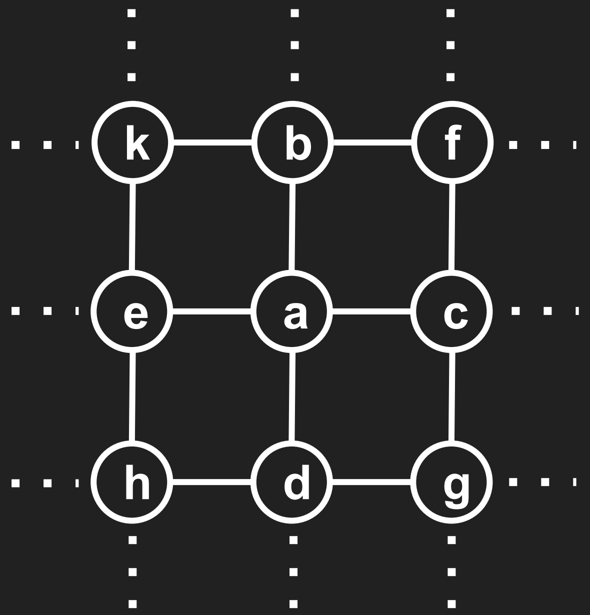


Example: Grids



$$P(X_1 \ldots X_n) = \frac{1}{Z} \prod_{i=1}^{n} \Phi_i(X_i) \prod_{(i,j) \in \mathcal{E}} \Phi_{ij}(X_i, X_j)$$

Example: Grids



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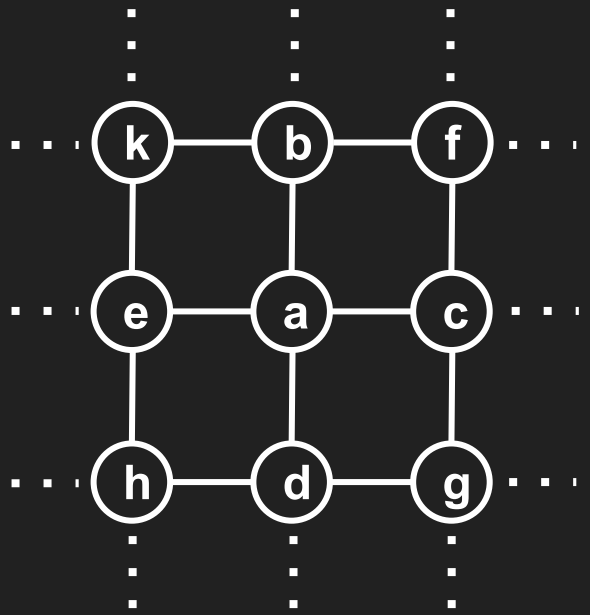

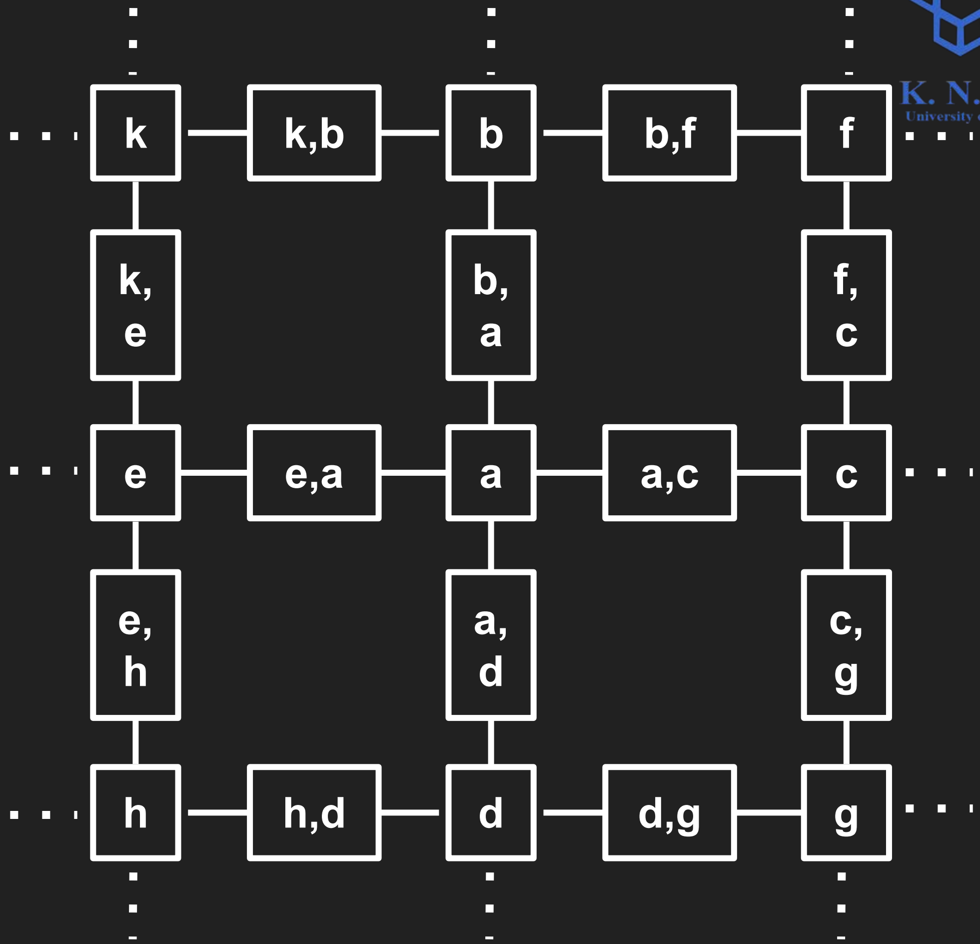

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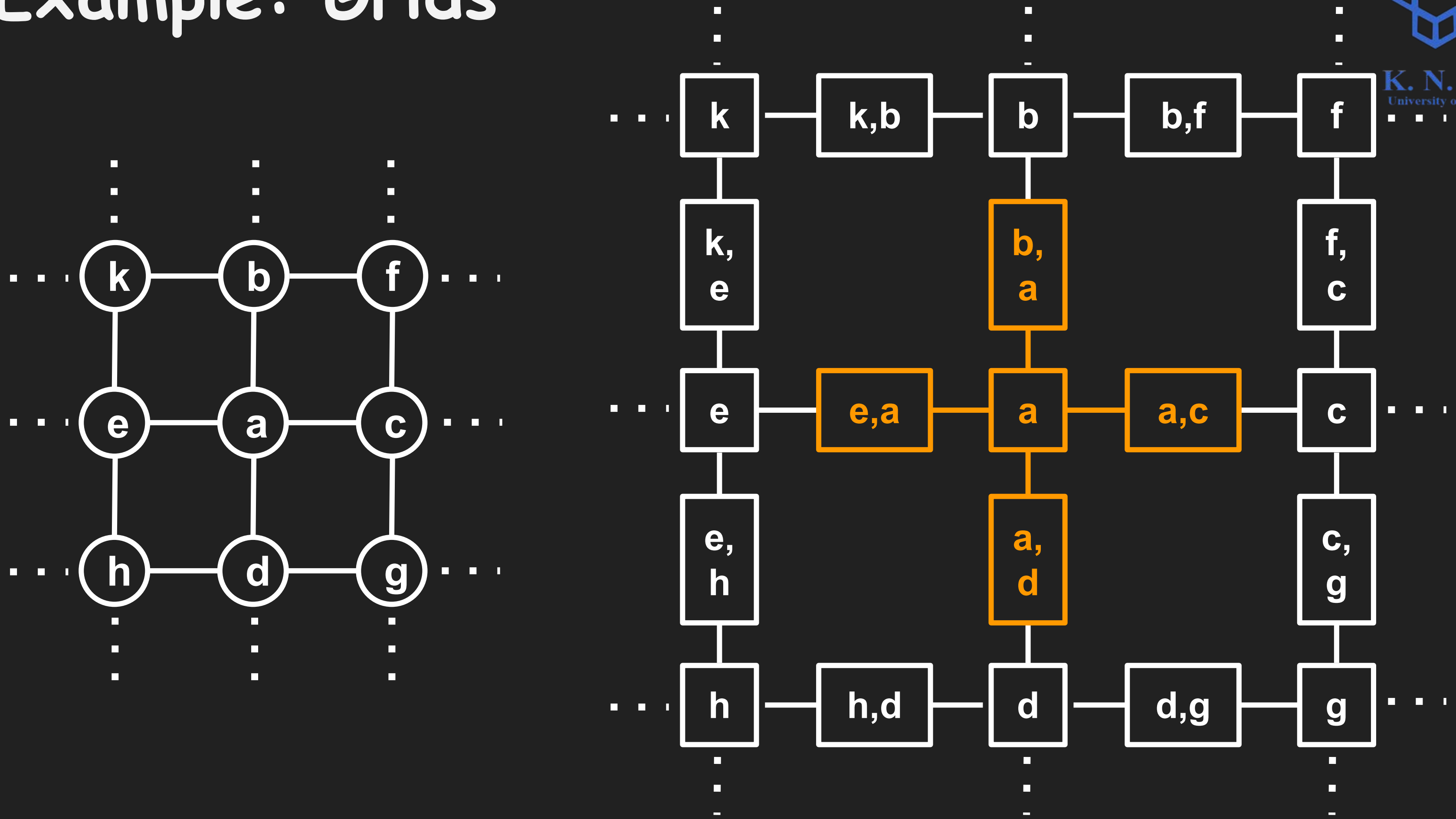

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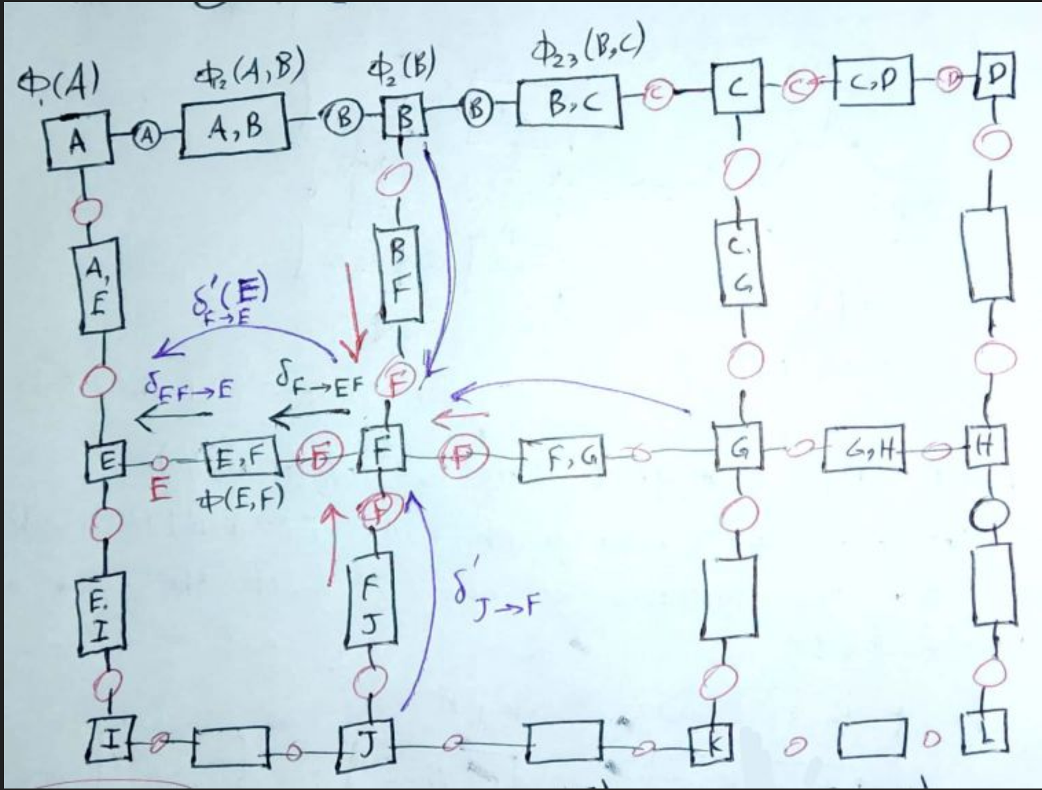

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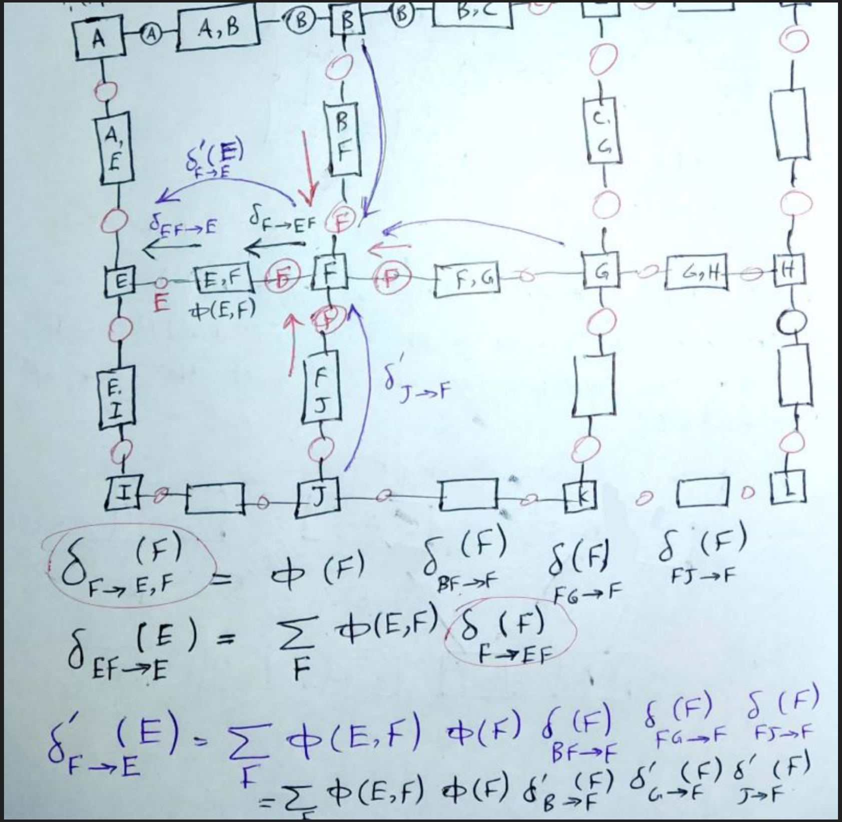

Calibration

Calibration and Convergence

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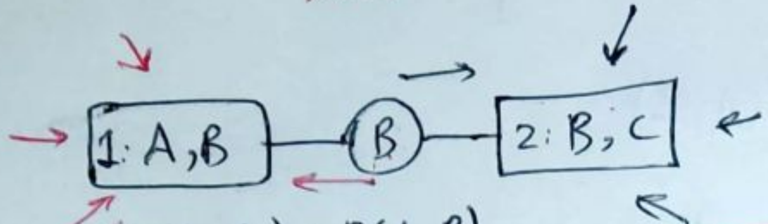
20 III

Assume that message passing gives the exact marginals $\beta(A,B) = P(A,B)$

Belief



فرض
$$\begin{cases} \beta(A,B) = P(A,B) \\ \beta(B,C) = P(B,C) \end{cases} \Longrightarrow \boxed{\sum_A \beta(A,B) = \sum_C \beta(B,C)}$$

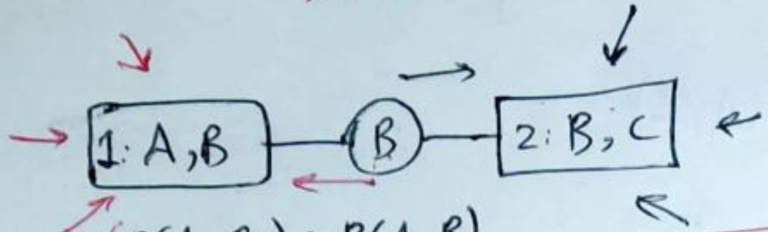
$$\boxed{P(B) = \sum_A P(A,B) = \sum_C P(B,C)}$$

Calibration and Convergence

20 III

Assume that message passing gives the exact marginals $\beta(A,B) = P(A,B)$

Belief



فرض

$$\begin{cases} \beta(A,B) = P(A,B) \\ \beta(B,C) = P(B,C) \end{cases} \Longrightarrow \boxed{\sum_A \beta(A,B) = \sum_C \beta(B,C)}$$

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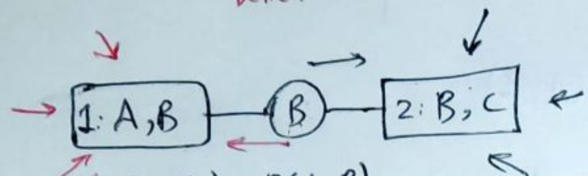
Calibration and Convergence



فرض
$$\begin{cases} \beta(A,B) = P(A,B) \\ \beta(B,C) = P(B,C) \end{cases} \Rightarrow \boxed{\sum_A \beta(A,B) = \sum_C \beta(B,C)}$$

$$\boxed{P(B) = \sum_A P(A,B) = \sum_C P(B,C)}$$

true solution $\Longrightarrow$ Beliefs agree upon the sepset (graph is calibrated)

$\Longleftarrow$? NOT Necessarily

Graph Calibrated $\Longrightarrow$ Message Passing has converged

Messages do not chang anymore.